The Role Of Virtual Manipulatives On The Concrete-Pictorial-Abstract Approach In Teaching Primary Mathematics

LEE Ngan Hoe
e-mail: nganhoe.lee@nie.edu.sg
National Institute of Education, Nanyang Technological University
Singapore

TAN Boon Leong Jeremy
e-mail: tan_boon_leong_jeremy@moe.edu.sg
Anchor Green Primary School, Ministry of Education
Singapore

Abstract

This paper reports on the work to review the key pedagogical approach advocated in Singapore Primary Mathematics Curriculum – the Concrete-Pictorial-Abstract Approach or C-P-A Approach in view of the changes brought about by technological advances. In particular, the Concrete-Virtual-Pictorial-Abstract Approach or C-V-P-A Approach is proposed to take into account the role that virtual manipulatives play in enriching the representations of mathematical concepts in the mathematics classrooms. Through a case study, the study sought to determine the possible impact of this proposed revised approach on teachers' delivery of lessons.

1. Introduction

The Singapore Mathematics Curriculum advocates the concrete-pictorial-abstract (C-P-A) development of concepts at the primary levels ([14], p.7). This approach is based on the work by Bruner ([2], p.10), who stressed that for full conceptual understanding, pupils translates "experiences into a model of the world" in three distinct ways – enactive, iconic and symbolic. He refers these three ways as modes of representation in which information are stored and encoded in memory. The enactive mode representation involves encoding "action based information" and storing it in the memory, while information is stored visually in the form of images during the iconic mode of representation. Finally, information is stored in the form of a code or symbol when the symbolic mode of representation is achieved. Bruner believes that a learner even of a very young age is capable of learning any material so long as the instruction is organised appropriately, following a progression from enactive to iconic to symbolic mode of representation.

In a subtly different but yet essentially similar manner, the Singapore C-P-A development of concepts approach encourages teachers to use multiple forms of representation to achieve the abstract conceptualisation of mathematics. In particular, it promotes the use of concrete, pictorial, and abstract modes of representation of a mathematics concept, such as place values or fractions, and the need for the learners to observe the equivalence of these representations to achieve sound conceptualisation of the abstract mathematics ([5], p.43).

With the advent of technology, the use of technological tools is also promoted ([14], p.19). Leung [12] stated the three essential features of technological tools as:

1. Efficiency in mathematics manipulation and communication
2. Multiple representation of mathematics, especially the efficient coupling of visual representation with other forms of representation
3. Interactivity between the learner and mathematics

Virtual manipulatives, one such type of technological tool, are virtual representations which can be manipulated dynamically the same way as corresponding concrete manipulatives to help develop mathematical conceptual understanding [17]. Ng ([19], p. 26) also call for the use of virtual manipulatives in the Singapore mathematics classroom to help "motivate and engage children in the learning process". In fact, Lee and Ferrucci [11] found that the use of virtual manipulatives not only engaged Singapore students in their learning of fractions, but also appeared to have a positive effect on narrowing students' achievement. However they also noted that there is a need to re-examine the C-P-A approach in the light of the availability of such pedagogical rich technological tool to help teachers better appreciate and integrate the use of virtual manipulatives into the mathematics classroom.

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2. The C-P-A Approach and Virtual Manipulatives

In examining the role of representations in the development of mathematical concept, Goldin and Shteingold [7] observed that much of "the history of mathematics is about creating and refining representational systems, and much of the teaching of mathematics is about students learning to work with them and solve problems with them". They pointed out that the representational systems important to mathematics and its learning have "structure", so that the different representations within a system are richly related to one another. A representation is "typically a sign or a configuration of signs, characters, or object", and they made a distinction between external systems of representation from the internal, psychological representational systems of individuals. They also observed that conceptual understanding "consists in the power and flexibility of the internal representations, including the richness of the relationships among different kinds of representation. It is through interaction with "structured external representations in the learning environment' that students' internal representational systems develop.

In advocating the use of the C-P-A development of concepts in the Singapore mathematics classrooms, Ng [19] advised teachers to "structure" the external representations in the learning environment, whenever possible, to enable students to progress from "concrete and pictorial levels to abstract representation". It is thus a common practice for teachers adopting the C-P-A approach to present abstract mathematical ideas using concrete, pictorial and abstract representations and encouraging students to establish linkages among these external representations to aid students in their development of their internal representational system of the abstract mathematical idea.

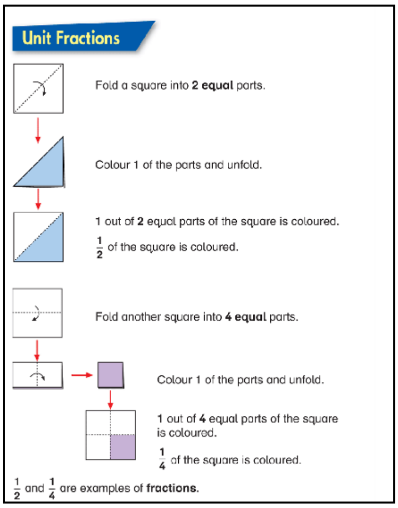

Figure 1: *using the C-P-A Approach ([4], p.70)*

Figure 1, for example, illustrate the use C-P-A approach to teach the concept of unit fractions. The teacher in this case uses three external representations, namely the concrete representation with the folded paper, a pictorial representation of the folded paper, and the abstract representation "$\frac{1}{4}$" to help students to develop an internal representation of a quarter. This, when coupled with more exposure to similarly structured external representations of other unit fractions or an external representation system of unit fractions, will eventually lead students to an evolvement of an internal representational system for unit fractions.

The works by Montessori [15], Piaget [20], and Bruner [2] are often cited by educational theorists as a theoretical basis for the use of concrete manipulatives. McNeil and Jarvin [13] observed that these early theorists believed that "children do not come into the world with the capacity for abstract thought" but "[I]nstead, they must construct abstract concepts through their interactions with concrete objects in the environment".

On the other hand, in his study of representation and understanding, Janvier [8] noted that conceptual understanding is a cumulative process mainly based upon the capacity of dealing with an "ever-enriching" set of representations. As the use of technology becomes more prevalent, especially in the Singapore context [11], teachers now have the opportunity to make use of another external representation through the electronic channel to promote conceptual understanding. In the studies by Lee and Chen [10] and Yuan, Lee and Wang [23], they advocated that the virtual manipulative as one such tool. Unlike static representations which are essentially picture, virtual manipulatives are dynamic representations that can be manipulated in the same way as concrete manipulatives [17]. Lee and Ferrucci [11], for example, reported the use of the virtual manipulative *Fractions* [3] to promote students' conceptual understanding of equivalent fractions. The virtual manipulative *Fractions*, though not a concrete manipulative, behaves much the same way as the concrete counterpart – the fraction discs, a commonly used concrete manipulative to teach fraction concepts at the primary school levels. It allows the user, with a few clicks of the mouse, to break two circular discs each into an equal number of parts to represent two different fractions, and then superimpose these two virtual representations to check for equivalence of the two corresponding fractions. The dynamism of the representation afforded by the virtual manipulative is certainly unlike a pictorial representation, which is basically static.

Clearly, the external representation that virtual manipulatives offer is one that is neither concrete nor pictorial, as proposed under the C-P-A approach. It is no wonder that teachers are unsure how virtual manipulatives affect the C-P-A development of concept, which is advocated at the primary levels in the Singapore context [11].

As Kaput [9] observed, there is a common agreement that mathematics teachers, not technological tools, are key change agents to bring about reform in mathematics teaching with technology. Moyer [16] also observed that "simply using manipulatives (concrete or virtual) is not enough if we do not consider how classroom teachers are using them". Stigler and Heibert ([22], p.75) pointed out that teaching is a system, and each feature by itself "doesn't say much about the kind teaching that is going on", and "[W]hat is important is how the features fit together to form a whole".

Moyer's [16] study on ten middle grade teachers found that "using manipulatives was little more than a diversion in classrooms where teachers were not able to represent mathematics concepts themselves". The teachers communicated to her that the manipulatives were fun, but not necessary, for teaching and learning mathematics. Even when teachers have participated in professional development in which manipulatives and technology were the major resources used, Moyer-Packenham, Salkind, Bolyard [18] found it was common for teachers to use the virtual manipulatives alone or to use physical manipulatives first, followed by virtual manipulatives. The researchers recognised that "physical manipulatives must also be manually linked to other representations, such as pictures or symbolic notations", much in agreement with the C-P-A approach of concept development. At the same time, they pointed out that the virtual manipulatives may only include connected representations in which the manipulation of one representation also produces a matched change in another representation. However, one should not leave it to chance for the pedagogical soundness of the C-P-A approach to get realised when the approach is combined with the use of virtual manipulatives. Teachers should be offered a revised C-P-A approach, taking into consideration the pedagogical richness of virtual manipulatives ([19], [21], and [11]), and without compromising the approach's power in establishing linkages among external representations to aid students in their development of conceptual understanding.

In fact, as early as 1960, Bruner ([1], p.82) in his examination of "aids for teaching", recognised that there are other more "subtle devices that can be and are being used to lead the student to a sense of the conceptual structure of things he observes". He felt that the best way to characterise them is to call them "sequential programs". Such a 'sequential program' can be observed in the C-P-A approach. The concrete, pictorial and abstract representations of an abstract mathematical idea are aids for teaching for the development of the concept associated with the mathematical idea. The sequence in introducing these external representations from concrete to pictorial to abstract ones helps the students to relate their concrete experiences with the abstract mathematical idea – narrowing the 'cognitive gap' that exists between the two. The use of each of these three external representations further narrow the 'cognitive gaps' that also exists between any two such representations in the sequence. As noted earlier, the external representation afforded by the virtual manipulatives lies between that of the concrete and pictorial. We could see the virtual manipulatives as another external representation that even further narrow the 'cognitive gap' between the concrete and pictorial representations. A possible revision to the C-P-A approach could be the C-V-P-A approach, where V refers to the external representation arising from the use of virtual manipulatives.

However, the C-V-P-A approach would require the teachers to use and make available the comparison of all these four representations. This could further discourage teachers' acceptance of such an approach when the use of concrete manipulatives has already contributed to sufficient mess and frustration due to the logistic involved in equipping the students with sufficient sets of these. Instead, we could propose a revision to the C-P-A approach to C-V and V-P-A approach. Fey [6] observed that the virtual manipulatives is a representation that plays "a role in helping move students from concrete thinking about an idea or procedure to an ultimately more powerful abstract symbolic form", suggesting the 'cognitive closeness' of the concrete and virtual representations. Thus, by presenting the revised approach as a two-part approach, it allows the teacher to use and encourage linking of the concrete and virtual representations to free both the teacher and students of the messy and frustrating logistic involved in using the concrete manipulatives. The teacher could then proceed to use and encourage comparison of the other three external representations in sequence, virtual, pictorial, and abstract, thus not forgoing the richness in using multiple external representations to promote conceptual understanding.

3. Methodology

This study seeks to determine the impact of the proposed revised pedagogical approach C-V & V-P-A towards concept development on teacher's planning and delivery of lessons.

The methodology adopted is a case study involving a participating mathematics teacher, Teacher X, teaching in a neighbourhood school at the Primary 3 level (8 to 9 years old). Teacher X is an experienced male teacher in-charge of the mathematics programme at the lower primary levels. He has a keen interest and attempted to use virtual manipulatives in his mathematics class, and expressed an interest in adopting a more prevalent use of virtual manipulatives in his mathematics teaching if there is a greater level of explicitness with regards to the role virtual manipulatives play in the advocated C-P-A approach. He believed that such a bigger picture of the key pedagogical approach in the teaching of mathematics could guide him better in adopting a sound pedagogy in delivering his mathematics lessons when considering the use of technology in the mathematics classroom.

In the study, he was given a short one-hour long professional development on the revised pedagogical approach C-V & V-P-A, and decided to use it as a guiding principle to plan a unit of lesson on "equivalent fractions" targeting at Primary 3 students. The content to be included in this unit on equivalent fractions is shown in Table 1. The pre-requisite knowledge on fractions that these Primary 3 students possessed are those taught in Primary 2 (Table 2).

Table 1: *content to be included in the unit on equivalent fractions ([14], p.39)*

Topics/Sub-topics	Content
Equivalent fractions	Include: • recognising and naming equivalent fractions, • listing the first 8 equivalent fractions of a given fraction, • writing the equivalent fraction of a fraction given the denominator or the numerator, • expressing a fraction in its simplest form, (Denominators of given fractions should not exceed 12.)

Table 2: *pre-requisite knowledge on fractions that Primary 3 students possessed ([14], pp.33-34)*

Topics/Sub-topics	Content

Fraction of a whole	Include:
	• interpretation of fraction as part of a whole,
	• reading and writing fractions,
	• comparing and ordering
	    ∗ unit fractions,
	    ∗ like fractions.
	
	(Denominators of given fractions should not exceed 12.)
	Exclude fraction of a set of objects.

Addition and subtraction of fractions	Include addition and subtraction of like fractions within one whole. (Denominators of given fractions should not exceed 12.)

The teacher decided to deliver the unit as a three one-hour lessons. The three lessons were designed with C-V & V-P-A approach as the basis to allow a more inductive approach to understanding the abstract concept of equivalent fractions. A relationship between the concrete and virtual manipulative was first established in the lesson plans to help pupils draw the link between the two representations. The development of conceptual understanding was then further strengthened by getting students to establish the equivalence of the virtual, pictorial and abstract representations in the lesson plans. A summary of the three lessons is provided in Table 3.

In the first part of Lesson 1 (Appendix A), the relationship between the concrete manipulative (Laminated circles) and the virtual manipulative *Fractions*[1] [3] , (C-V) was first established through teacher demonstration. In the development part of Lesson 1, the concept of equivalent fractions was established through the C-V & V-P-A approach whereby students were able to investigate by listing equivalent fractions using the concrete manipulative Laminated Circles

[1] Access to a copy of the virtual manipulative *Fractions* is available at Lee & Ferrucci (2012).

while the teacher reinforce the links between the various external representations using the virtual manipulative *Fractions.*

Table 3: *summary of the three lessons*

Lesson Number	Lesson Objectives	Remarks
1	Recognizing and naming equivalent fractions.	Use the C-V & V-P-A Approach to develop the concept of equivalent fractions.
2	Listing of the first 8 equivalent fractions of a given fraction. Writing the equivalent fraction of a fraction given the denominator or the numerator	Students given a choice of C-P-A and V-P-A to list fractions and induced the rule that exists between equivalent fractions.
3	Expressing a fraction in its simplest form.	Tap on pupils' conceptual understanding of equivalent fractions to introduce the idea of expressing a fraction in its simplest form.

In Lesson 2 (Appendix B), with the aid of the virtual manipulative *Fractions*, the V-P-A approach was used to further strengthen the conceptual understanding of equivalent fractions by getting pupils to list out eight equivalent fractions of a given fraction. Students could either choose to use the virtual manipulative *Fractions* or the concrete manipulative Laminated Circles to complete the task. The concrete manipulative was provided for the benefit of students who may still choose to work on these instead of the virtual ones. Students were then encouraged to examine for patterns and to predict a possible rule that governs the numerators and denominators in the listing of equivalent fractions found. The rule between the numerators and denominators of equivalent fractions was finally established through a guided process of induction. Finally, students were guided to apply the rule to determine missing numerator or denominator of equivalent fractions. As the second and final parts of Lesson 2 assumed an established conceptual understanding of equivalent fractions, i.e. students would have already established the abstract internal representation system for equivalent fractions, there was no explicit planned teaching action involving the C-V & V-P-A for these two parts of the Lesson. However, if the need arose, such as when weaker students were observed to be lacking in arriving at the abstract level of conception, the approach would be called into play to assist such students in their learning process. Teacher X pointed out, though, that in such a situation, he would likely be using the approach in reverse order. In other words, he would first link the abstract concept to a pictorial representation, and then successively to virtual and concrete representations only if links between earlier representations failed.

Lesson 3 (Appendix C), the final of the three lessons, introduced the students to the idea of expressing a fraction in its simplest form. This is achieved through getting students to compare and examine the different symbolic equivalent representations of a given fractions. Consequently, as in the second and third parts of Lesson 2, consequently students are assumed to have established a conceptual understanding of equivalent fractions. Thus, there is no explicit planned teaching action involving the C-V & V-P-A for Lesson 3. And, as in the case of Lesson 2, Teacher X has put in place similar remedial teaching actions to help students who may have not achieved the desired level of conception of equivalent fractions.

At the end of the three lessons, an interview was conducted on Teacher X to obtain his feedback on the impact of the C-V & V-P-A Approach on his planning and delivery of the lessons.

4. Feedback from Teacher X

Based on an one-hour interview from the Teacher X, the feedback from him on the impact of the C-V & V-P-A Approach on his planning and delivery of the lessons could be classified into four key areas.

4.1 An Integrated pedagogical approach

Teacher X pointed out that using virtual manipulatives as another external representation of mathematical concepts in the modified C-V & V-P-A Approach helped to provide an effective coupling between the C- and P- representations. He felt that it provided a formal and efficient way to view how virtual manipulative could be integrated with the C-P-A approach. Clearly, Teacher X also viewed the virtual manipulative as a narrowing of the cognitive gap between the concrete and pictorial representations.

4.2 An improvement in the efficiency of lessons

As the modified C-V & V-P-A Approach allows the teacher to proceed to just using virtual, pictorial, and abstract representations for development of concept once students are able to see the equivalence between the concrete and virtual representations through a teacher-led class discussion, Teacher X was glad that it helped to free him from the logistic and time needed to provide, distribute and retrieve the concrete manipulatives. He felt that the modified approach has increased the efficiency of his lesson delivery.

4.3 An improvement in the effectiveness of lessons

Teacher X observed that the virtual manipulative was a very effective learning tool to increase the engagement of students during lessons. As students nowadays are mostly digital natives, meaning that most of them are very much exposed to digital resources and multimedia in their lives, the students were observed to be more comfortable and enthusiastic with the use of the virtual manipulatives than the concrete one. This was reflected by the response of students to the question in Part B of Activity Sheet 2 (Appendix E) during Lesson 2, whereby students were asked to indicate explicitly their choice between using concrete or virtual manipulatives to determine the equivalent fractions of $\frac{1}{3}$. Other than the few students who did not attempt to respond to the question, all the remaining students, which is more than 80% of the students in the class, reflected that the virtual manipulative was their choice for the activity. In fact, Teacher X shared that he observed that there was an increased level of comfort and engagement among the students when using the virtual manipulative compared to past experience when he only used the concrete manipulative under the C-P-A Approach. He also noted that students were quicker to engage themselves in higher order thinking without being prompted. Though the derivation of the rule for governing the numerators and denominators of the equivalent fractions was to be covered in Lesson 2, a number of students were observed to be already engaged in the inductive process at the end of Lesson 1 when they were investigating for equivalent fractions of $\frac{1}{2}$. One student, for example, asked the teacher if she could write down some of her answers without using the virtual manipulative as she said that she thought “she has already found the pattern!”. This is in agreement with the findings of Lee and Ferrucci’s [11] study; they found that by allowing students to manipulate the on-screen objects to test hypotheses and experiment ideas, the experience enhances students’ thinking and creativity.

4.4 An equity issue

Teacher X's only concern with the modified C-V & V-P-A Approach was one related to equity issue. He believed that with the modified Approach, which contributed to both the efficiency and effectiveness of the delivery of mathematics lessons, the use of technological tools may become more pervasive in the mathematics classrooms. However, he observed that more of the students in higher ability grouping were observed to be more at ease with the use of virtual manipulative than those in the lower ability group. He attributed it to the possible issue of equity as his informal survey revealed that students in the higher ability grouping have greater level of access to computers in their home environment. He felt that though the school has provided all students with access to computers or laptops during lesson time, the difference in exposure to technological tools beyond curriculum time may have contributed to differences in the level of confidence and ease when manipulating technological tools for the purpose of learning due to practice effect.

5. Discussion and Conclusion

As the use of technological tools becomes more prevalent, a reasonable concern would be possible greater divide between the have's and have not's. It is crucial and urgent that the issue of equity be appropriately addressed as Singapore pride itself to be a meritocratic system. In fact, Teacher X was quick to add that schools are looking into improving access of technological tools for learning beyond curriculum time.

Nonetheless and despite the limitation for generalisation of the study due to the fact that it is only a case-study, the modified C-V & V-P-A Approach appeared to be able to help teachers to better perceive the role of virtual manipulatives – a technological tool, within the context of the commonly used C-P-A Approach used in the Singapore Mathematics Classrooms. The adaptation is viewed more as an integrative rather than additive approach to new pedagogical tools; the adaptation does not simply appear to add value, it is also being perceived to be removing certain concern, resulting in an increased effectiveness and efficiency of lesson delivery. The C-V & V-P-A Approach seems to have helped teachers to structure the external representations in the learning environment as proposed by Golden and Shteingold [7]. The Approach encapsulated the idea suggested by Janvier [8] that conceptual understanding is a cumulative process in developing students' capacity in dealing with an ever-enriching set of representations, including virtual manipulatives. At the same time, the Approach appears to allow teachers to see how the use of virtual manipulatives fit into the bigger picture of the C-P-A approach advocated in the national curriculum – an important feature to ensure a greater level of success when introducing new pedagogical features into a teaching system, as observed by Stigler and Heibert [22].

A more extensive study that involves not only more teachers but also applying the modified C-V & V-P-A Approach across the teaching of different topics is needed to fully evaluate the soundness of this modified approach to replace the C-P-A Approach. However, there are also other related issues that may be worthwhile for further investigation:

1. Can the C-V & V-P-A Approach be further refined to help teachers close the 'cognitive gaps' between the various representations to better promote conceptual understanding in the mathematics classrooms without losing the generosity and practicality of such an approach? Could other technological tools other than virtual manipulatives, further contribute to such a refinement?
2. External representations are ways that teachers, based on their pedagogical content knowledge, introduce in their teaching acts to help students to acquire the necessary level of internal representation, or some refer to as abstraction [20]. Are there corresponding levels of mathematical levels of abstraction, and if there are, how are these related with the different types of external representations?

Insights to the above two questions would certainly improve the process of teaching and learning as the gaps between external and internal representation systems could then be better addressed.

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Appendix A

Lesson 1

Name of Teacher:	Teacher X	**Subject:**	Mathematics	**Topic:**	Equivalent fractions
Objectives:	**Specific Instructional Objective** At the end of the lesson, pupils will be able to recognize and name equivalent fractions				
Difficult Concepts:	Fixed quantity can have multiple names for fractions			**Duration:**	1 hour (60 min)

	Key ideas/ concepts to be developed	Sequence of Development (Instructional Procedure and Learning Styles Strategies utilised)	Strategies/ Rationale	Resources
15 min	**Activating of pre-requisite knowledge on fractions**	**Introduction:** **Think-pair-share** - Students pair up with their shoulder partner. - Each student will be given a laminated circle paper. - Student to draw a fraction on the circular laminated piece and get their partner to write down the fraction that they had drawn.		Laminated Circle paper (there will be two sets: small for students, big for teacher to illustrate on the board)
30 min	**Introduction to concept of equivalent fraction**	**Development** **Activity 1: Divide and divide activity** - Students will work in pairs. Students will be numbered "1" and "2" - Using the same laminated circle paper, teacher to illustrate a fraction $\frac{1}{2}$ by shading a part of the circle. - Students to draw the same fraction on their own circle. (Student will shade the part) - Teacher show the same fraction using the virtual manipulative Fractions. - Teacher tells Student 1 of each pair to cut the circle into 4 equal parts now. - Student 1 of each pair will have to write down the fraction of the shaded part, i.e. students should write "$\frac{2}{4}$" . - Teacher to illustrate using virtual manipulative. - Student 2 of each pair to check their partner's answer. - Teacher then tells Student 2 of each pair to cut the circle into 8 equal parts.	**C-V of the C-V & V-P-A Approach**	- Laminated Circle paper - Activity Sheet 1 (Appendix D) - Whiteboard Markers - Virtual manipulative *Fractions* - Whiteboard

	Key ideas/ concepts to be developed	**Sequence of Development** (Instructional Procedure and Learning Styles Strategies utilised)	**Strategies/ Rationale**	**Resources**
15 min		- Student 2 of each pair will have to write Down the fraction of the shaded part now, i.e. students should write "$\frac{4}{8}$" . - Teacher to illustrate using virtual manipulative. - Student 1 of each pair to check their partner's answer. - Get students to compare and conclude that the listed fractions are equivalent. - Teacher to explain that the word "equivalent" is derived from "equal" and as the fractions are equal, they are called "equivalent fractions" - Teacher get students to work in pairs and to list out another 4 equivalent fractions of "$\frac{1}{2}$" in Activity sheet 1. (Pupils can choose between using the concrete manipulative or the virtual manipulative.) **<u>Consolidation and Closure</u>** - Elicit response from students with regards to the tools which they chosen to use to help them in the previous activity. - Teacher to reinforce on the key concept that "Equivalent" comes from the word "equal". Thus although there were many different ways in which a part of a whole can be described, they all meant the same part. - Pupils to complete journal (Journal Worksheet 1) individually		Activity Sheet 1 (Appendix D) Journal Worksheet (Appendix G)

Appendix B

Lesson 2

Name of Teacher:	Teacher X	**Subject:**	Mathematics	**Topic:**	Equivalent fractions
Objectives:	**<u>Specific Instructional Objective</u>** At the end of the lesson, pupils will be able to - List the first 8 equivalent fractions of a given fraction - Write the equivalent fraction of a fraction given the denominator or the numerator				
Difficult Concepts:	Fixed quantity can have multiple names for fractions			**Duration:**	1 hour (60 min)

	Key ideas/ concepts to be developed	**Sequence of Development** (Instructional Procedure and Learning Styles Strategies utilised)	**Strategies/ Rationale**	**Resources**
10 min	**Activating of pre-requisite knowledge on fractions**	**<u>Introduction:</u>** **<u>Think-pair-share</u>** - Students pair up with their shoulder partner. - Each student will be given laminated circles - Student is to draw a fraction and get their partner to write down an equivalent fraction to the fraction that they had drawn.		Laminated circles
20 min	**Concept of equivalent fraction**	**<u>Development</u>** **<u>Activity 1: Create the equivalent fractions</u>** - Students are to find 8 equivalent fractions of a given fraction using the virtual manipulative *Fractions* or the concrete manipulative. - Teacher to observe the working of pupils. Give help to weaker pupils.		- Virtual manipulative *Fractions* - Concrete manipulative: Fraction discs - Activity Sheet 2 – Parts A & B (Appendix E)
15 min		**<u>Consolidation and evaluation</u>** - Teacher will elicit response from students on the following: 1. Which tool did you use to help you in your task? 2. How did you create the equivalent fractions? - Teacher gets students to work in pairs to look at the 2 sets of equivalent fractions and discuss about the patterns which they can see. - Teacher lead students to see that the numerator and denominator of the fraction increases or decrease by the same factor. - Teacher to illustrate that a missing numerator or denominator can be found using the rule that governs the numerator and denominator of "Equivalent Fractions".	**V-P-A of the C-V & V-P-A Approach**	

	Key ideas/ concepts to be developed	**Sequence of Development** (Instructional Procedure and Learning Styles Strategies utilised)	**Strategies/ Rationale**	**Resources**
15 min		**Closure:** - Teacher to reinforce on the pattern which was observed in the derivation of the equivalent fractions of a given fractions. - Students to complete journal individually.		Journal Worksheet (Appendix G)

Appendix C

Lesson 3

Name of Teacher:	Teacher X	**Subject:**	Mathematics	**Topic:**	Equivalent fractions
Objectives:	**Specific Instructional Objective** At the end of the lesson, pupils will be able to express a fraction in its simplest form				
Difficult Concepts:	Fixed quantity can have multiple names for fractions			**Duration:**	1 hour (60 min)

	Key ideas/ concepts to be developed	**Sequence of Development** (Instructional Procedure and Learning Styles Strategies utilised)	**Strategies/ Rationale**	**Resources**
10 min	**Activating of pre-requisite knowledge on fractions**	**Introduction:** **Think-pair-share** - Students pair up with their shoulder partner. - Each student to share with their partner about the equivalent fractions which they found in Part C of Activity Sheet 2.		Activity Sheet 2 (Appendix E)
25 min	**Idea of expressing a fraction in its simplest form**	**Development** - Teacher to elicit a few responses from students with regards to the equivalent fractions they created in Part C of Activity Sheet 2. - Teacher to present their solutions on the whiteboard.		Activity Sheet 2 (Appendix E)
10 min		- Teacher to ask students to observe what is written on the whiteboard and to discuss with their partner again on what patterns they observed. - Students are to share their partner's observations - Teacher to explain the pattern in the numerator and denominator in each series of equivalent fractions. The numerator and denominator increase or decrease by the same factor. - Teacher reinforce on the concept of equivalent fractions. - Teacher to explain that the equivalent fraction with the smallest numerator/denominator is the fraction in the simplest form		
10 min		**Consolidation and evaluation** - Pupils to list out the equivalent fractions of another fraction without the manipulatives and to identify the fraction in its simplest form.		Activity Sheet 3 (Appendix F)

	Key ideas/ concepts to be developed	**Sequence of Development** (Instructional Procedure and Learning Styles Strategies utilised)	**Strategies/ Rationale**	**Resources**
15 min		**<u>Closure:</u>** - Teacher reinforce on the understanding of simplest form. - Pupils to complete journal individually.		Journal Worksheet (Appendix G)

Appendix D

Activity Sheet 1

Name:_________________ **Class:_________________**

$$\frac{1}{2} = \frac{\ }{\ } = \frac{\ }{\ } = \frac{\ }{\ } = \frac{\ }{\ } = \frac{\ }{\ } = \frac{\ }{\ }$$

Appendix E

Part A: Given the fraction on the left, create 8 equivalent fractions using either the fraction disc provided or *Fractions* on the computer.

$\frac{1}{3}$								

Part B: Answer the following questions

I am ____________________ (name) from ______________ (class)

I chose the fraction disc / computer * to help me in my activity

*delete one accordingly

I finish my activity in _______ minutes

Part C: Create 8 equivalent fractions of the given fraction without the help of any tool.

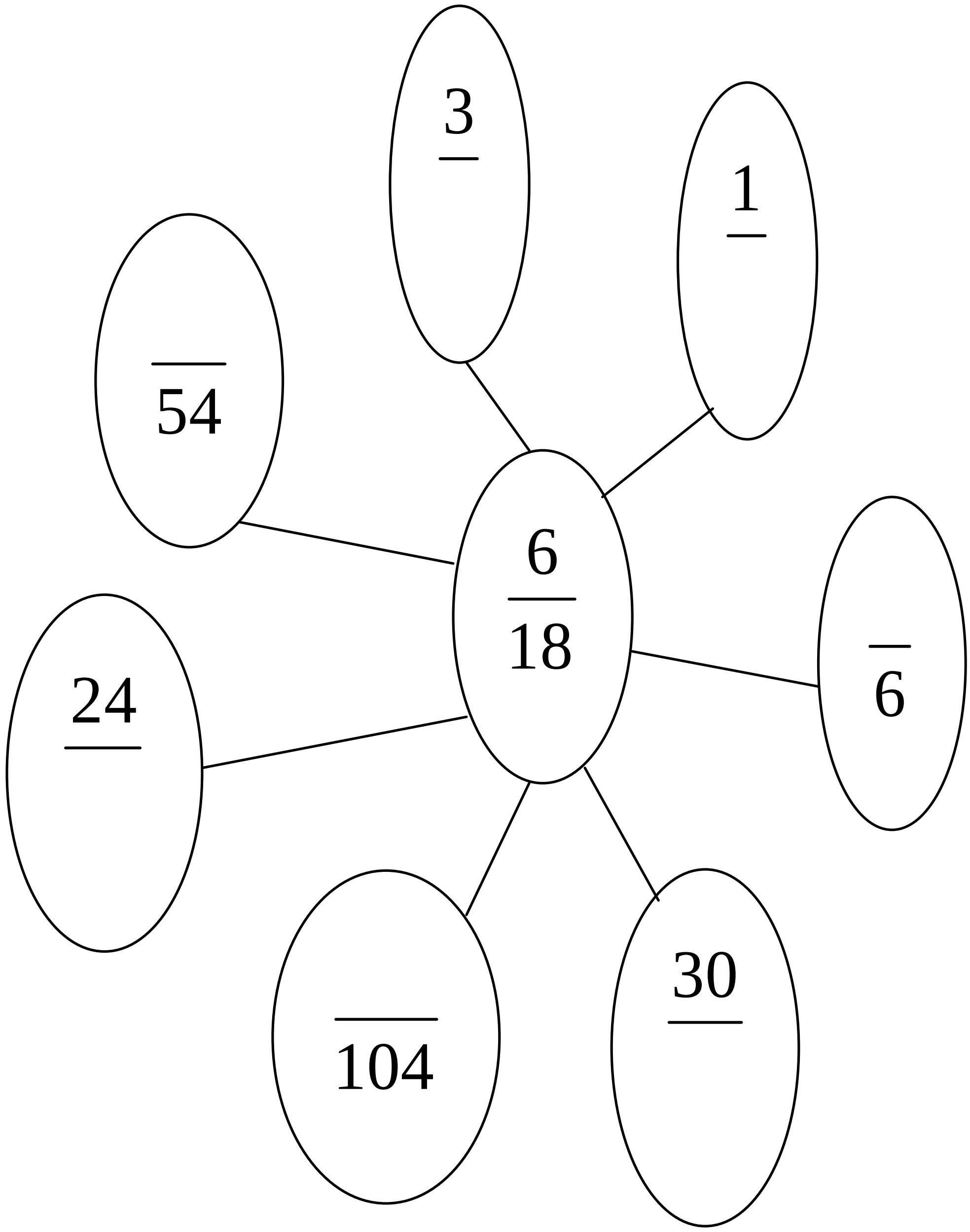
$\frac{1}{7}$								

Appendix F

Name: ______________________ Class: ___________________

Find the equivalent fractions of the fraction in the middle.

Colour the fraction in the simplest form.



Appendix G

My Math Learning Journal

Write a letter to your friend to tell him or her about the math lesson you had today in school.

Dear ______________ ,

My name is ____________________________ and I am from Class _________________

Today' s date is ____________________ .